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1976 J. Phys. A: Math. Gen. 9 1109

(http://iopscience.iop.org/0305-4470/9/7/014)

J. Phys. A: Math. Gen., Vol. 9, No. 7, 1976. Printed in Great Britain. © 1976

Percolation processes in two dimensions V. The exponent δ_p and scaling theory

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Received 12 January 1976

Abstract. By introducing a notional field variable λ into the percolation problem, a function $P_c(\lambda)$ is defined whose Ising analogue is the magnetic field variation of the magnetization along the critical isotherm. Series expansions are used to study the critical behaviour of $P_c(\lambda)$, characterized by an exponent δ_p, for both site and bond percolation problems on the more common two-dimensional lattices. We conclude that δ_p is a dimensional invariant and estimate $\delta_p = 18{\cdot}0 \pm 0{\cdot}75$.

It appears that $\delta_p = 18$, $\gamma_p = 2\frac{3}{7}$, $\beta_p = \frac{1}{7}$ is the simplest set of rational exponents which is most consistent with the available data and which satisfies the scaling law $\gamma_p = \beta_p(\delta_p - 1)$ exactly.

1. Introduction

In this paper we report numerical studies on the critical exponent δ_p (defined below) for bond and site percolation processes on various two-dimensional lattices and compare the result with that obtained from the scaling laws (Kasteleyn and Fortuin 1969, Essam and Gwilym 1971, Essam 1972, Dunn *et al* 1975a). There is a close formal analogy (Kasteleyn and Fortuin 1969, Essam 1972) between percolation processes and the ferromagnetic Ising model and we assume a general familiarity with both these problems such as may be derived from the reviews by Shante and Kirkpatrick (1971) and Essam (1972) for the one and by Fisher (1967) and Domb (1974) for the other. We have introduced the two-dimensional problem, defined the notation, derived new data and analysed series expansions for the mean cluster size and percolation probability in previous papers (Sykes and Glen 1976, Sykes *et al* 1976a, b, c referred to as I, II, III, IV respectively).

As the notation suggests, the percolation exponent δ_p will be defined in such a way that its ferromagnetic analogue δ describes the shape of the critical isotherm (Gaunt 1967, Gaunt and Sykes 1972). More specifically, we begin with the expansion for the mean number of finite clusters which, following I, can be written

$$K(p, \lambda) = \sum_{s=1}^{\infty} D_s(q) p^s \lambda^s \tag{1.1}$$

where p is the probability of occupation of the lattice sites (site problem) or bonds (bond problem), $q = 1 - p$ and $\lambda = e^{-\xi}$ is a notional field variable normally set to $\lambda = 1$ corresponding to zero field ($\xi = 0$). (Alternatively, a field variable may be introduced in a quite natural way by considering percolation theory as the low temperature limit of a dilute Ising model in an external magnetic field (Dunn *et al* 1975a).) D_s is a polynomial in q, the coefficient of q^t being the number of clusters per site or bond of the lattice of size s with perimeter t. The analogous expansion for a ferromagnetic Ising model is the high-field expansion of the configurational free energy (Sykes *et al* 1965)

$$\ln \Lambda = \sum_{s=1}^{\infty} L_s(u)\mu^s, \tag{1.2}$$

where u and μ are the usual low temperature and magnetic field variables, respectively. Here the coefficient of u^t in the polynomial $L_s(u)$ arises from configurations of s overturned spins having Ising perimeter t. Evidently, λ is the analogue of μ (see also III) and p the analogue of u or the temperature T, with $p < p_c$ equivalent to $T > T_c$, and vice versa. Furthermore, it is clear on considering the appropriate derivatives of (1.1) and (1.2) with respect to λ and μ, respectively, and then setting $\lambda = 1$ and $\mu = 1$, that the mean size of finite clusters and the percolation probability are analogous to the initial susceptibility and spontaneous magnetization, respectively, of the ferromagnet. For example, evaluating the magnetization (Sykes *et al* 1965)

$$M(u, \mu) = 1 - 2\mu(\partial \ln \Lambda/\partial\mu) \tag{1.3}$$

at $\mu = 1$ gives the spontaneous magnetization, while evaluating

$$P(p, \lambda) = 1 - \frac{1}{p}\lambda\frac{\partial}{\partial\lambda}K(p, \lambda) \tag{1.4}$$

at $\lambda = 1$ gives the percolation probability (Essam and Gwilym 1971). To define the exponent δ in the ferromagnetic case, the magnetization must be evaluated along the critical isotherm $u = u_c$, the basic μ-expansion (Gaunt and Sykes 1972)

$$M_c(\mu) \equiv M(u_c, \mu) = 1 - 2\sum_{s=1}^{\infty} sL_s(u_c)\mu^s \tag{1.5}$$

following from (1.2) and (1.3). At the critical point, it is assumed that

$$M_c(\mu) \sim E(1-\mu)^{1/\delta}, \qquad (T = T_c, \mu \to 1-), \tag{1.6}$$

to leading asymptotic order, which defines the critical exponent δ and critical amplitude E. Treating the percolation problem in an analogous way involves the evaluation of $P(p, \lambda)$ at $p = p_c$, (1.1) and (1.4) giving the basic λ-expansion

$$P_c(\lambda) \equiv P(p_c, \lambda) = 1 - \sum_{s=1}^{\infty} sD_s(q_c)p_c^{s-1}\lambda^s, \tag{1.7}$$

which is assumed to exhibit a dominant critical point singularity of the form

$$P_c(\lambda) \sim E_p(1-\lambda)^{1/\delta_p}, \qquad (p = p_c, \lambda \to 1-). \tag{1.8}$$

To derive the expansion (1.7) correct to order λ^N requires a knowledge of p_c and the perimeter polynomials $D_1, D_2, \ldots D_N$. These polynomials were derived in I, and are known through $N = 9$ T(B), 13 SQ(B), 17 HC(B), 14 T(S), 17 SQ(S), 11 SQM(S), 20 HC(S) and 9 HCM(S). For all the bond problems as well as the T(S) problem, we have used the exact value of p_c (Sykes and Essam 1964); for the site problems on the square and honeycomb lattices and their corresponding matching lattices, the best numerical estimate of II has been used.

2. Series analysis

All the standard techniques of series analysis reviewed by Gaunt and Guttmann (1974) have been employed in studying $P_c(\lambda)$. The best results are those obtained from the expansion coefficients m_n of the logarithmic derivative $-\lambda(d/d\lambda)\ln P_c(\lambda)$, which according to (1.8) should approach $1/\delta_p$ as $n \to \infty$. (Note that the analogous method of analysing the corresponding Ising problem proves to be the best in that case also (Gaunt and Sykes 1972).) Extrapolating these against $1/n$ gives the 'appropriate extrapolants' e_n which are plotted against n in figure 1. The 'appropriate extrapolants' have been defined previously (Gaunt and Sykes 1972); in most cases they are simply the linear

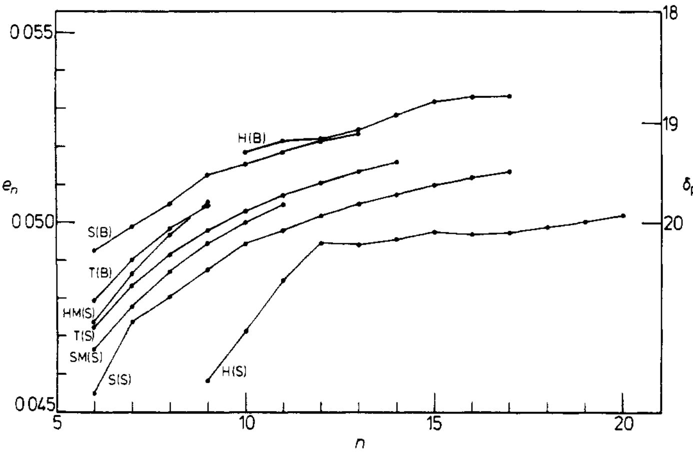

Figure 1. Appropriate extrapolants e_n plotted against n.

intercepts calculated from adjacent points but for the HC(B) and HC(S) problems which exhibit apparent periodicities of 4 and 3, respectively, the oscillations are smoothed-out by a suitable averaging procedure. The uncertainty in p_c (when the exact value is not available) introduces an uncertainty in the last extrapolant of about $\pm 0{\cdot}005$, corresponding to an uncertainty in δ_p of about ± 2.

The extrapolants appear to be increasing slowly and monotonically as n increases and assuming this trend continues the last point on each curve provides an upper bound on δ_p for that particular problem. If we further assume, as seems reasonable, that δ_p is a dimensional invariant then the plots must have a common limit and from the HC(B) problem we obtain our best upper bound of $\delta_p < 18{\cdot}75$. From an essentially subjective assessment of the rate of increase of the extrapolants, we think a limit close to $\delta_p = 18$ does not seem unreasonable. Accordingly we adopt as our best estimate

$$\delta_p = 18{\cdot}0 \pm 0{\cdot}75. \tag{2.1}$$

With regard to alternative methods of analysis, Padé approximants to the series for $(\lambda - 1)(d/d\lambda)\ln P_c(\lambda)$ evaluated at $\lambda = 1$ provide further support for a value in this vicinity. The sequences in table 1 are for the T(B) and T(S) problems and are fairly typical. The estimates appear to be converging from above to a limit close to (2.1). The ratio method is also in general agreement with these conclusions although convergence appears to be disappointingly slow.

Table 1. Estimates of δ_p for the T(B) and T(S) problems provided by the $[n+j/n]$ Padé approximants to the $(\lambda-1)(d/d\lambda)\ln P_c(\lambda)$ series evaluated at $\lambda=1$.

	T(B)			T(S)		
n	$[n-1/n]$	$[n/n]$	$[n+1/n]$	$[n-1/n]$	$[n/n]$	$[n+1/n]$
1	32·56	13·79	26·06	31·00	24·76	25·67
2	23·41	24·16	23·32	25·59	25·05‡	22·93
3	23·83	57·41†	19·09	20·84	21·93	20·86
4	19·74	22·77		21·57	20·80	20·89†
5				20·54	18·42	19·95
6				19·81	19·97	19·95‡
7				19·86‡		

† Defect on positive axis. ‡ Defect on negative axis.

To study the relatively slow convergence further, we have formed Padé approximants to the $(d/d\lambda)\ln P_c(\lambda)$ series. These reveal on the real axis for $\lambda>1$, a pole-zero sequence characteristic of a coincident singularity modifying the dominant asymptotic behaviour (1.8) to

$$P_c(\lambda)\sim E_p(1-\lambda)^{1/\delta_p}[1-F_p(1-\lambda)^{g_p}], \qquad (g_p>0). \tag{2.2}$$

An analogous modification of (1.6) occurs for the simple Ising ferromagnet (Gaunt and Sykes 1972). It is to be supposed that this presumed confluence is responsible in general for the slow rate of convergence, since with the exception of the HC(B) and HC(S) problems, no other singularities appear anywhere in the complex λ-plane. (For the HC(B) and HC(S) problems, we find non-physical singularities in the complex plane further from the origin than the physical singularity at $\lambda=1$. These singularities cause the oscillations mentioned earlier, which are smoothed-out by taking the 'appropriate extrapolants'.)

The exponent g_p may be estimated by returning to the series coefficients m_n, which as we have seen approach $1/\delta_p$ as $n\to\infty$. The rate of approach depends on the confluent singularity (see Gaunt and Sykes 1972). It follows from (2.2) that

$$m_n\sim\frac{1}{\delta_p}-\frac{A_p}{n^{g_p}}, \qquad (n\to\infty), \tag{2.3}$$

where

$$A_p=g_pF_p/\Gamma(1-g_p) \tag{2.4}$$

and $\Gamma(x)$ is the gamma function. Taking $\delta_p=18$, a sequence of estimates for g_p and A_p can be obtained by fitting (2.3) to successive pairs of coefficients m_n and m_{n-1}. These are presented in table 2 for all the bond problems and the T(S) problem. Corresponding sequences for the other problems are not very useful since large uncertainties are introduced through uncertainties in p_c and hence in m_n. On plotting the sequences for g_p in table 2 against n and assuming they have a common limit, a value around

$$g_p=0{\cdot}75\pm0{\cdot}05 \tag{2.5}$$

is suggested. Similarly, for A_p we estimate

$$A_p=0{\cdot}06\pm0{\cdot}01 \tag{2.6}$$

Table 2. Successive estimates for g_p and A_p assuming $\delta_p = 18$.

	g_p				A_p			
n	T(B)	SQ(B)	HC(B)	T(S)	T(B)	SQ(B)	HC(B)	T(S)
2	0·2925	0·3251	0·5017	0·3251	0·0415	0·0399	0·0410	0·0399
3	0·5364	0·4612	0·5070	0·4612	0·0492	0·0439	0·0412	0·0439
4	0·5513	0·6840	0·2329	0·4939	0·0500	0·0561	0·0305	0·0455
5	0·6040	0·6104	0·6794	0·5451	0·0538	0·0506	0·0566	0·0488
6	0·6040	0·6477	1·2431	0·5680	0·0538	0·0537	0·1401	0·0507
7	0·6231	0·6465	0·9518	0·5877	0·0556	0·0536	0·0831	0·0525
8	0·6397	0·6540	0·3315	0·6022	0·0575	0·0544	0·0249	0·0540
9	0·6506	0·6804	0·4848	0·6137	0·0588	0·0575	0·0342	0·0553
10		0·6767	0·9835	0·6227		0·0570	0·1023	0·0564
11		0·6832	0·9372	0·6299		0·0579	0·0920	0·0573
12		0·6886	0·6124	0·6359		0·0586	0·0422	0·0582
13		0·6876	0·6362	0·6407		0·0585	0·0448	0·0589
14			0·8034	0·6448			0·0688	0·0595
15			0·8042				0·0689	
16			0·7290				0·0562	
17			0·7186				0·0546	

in all cases; A_p is not expected to be a dimensional invariant but the sequences are not sufficiently regular for us to discern the relatively small differences between problems. Using (2.5) and (2.6) to calculate F_p from (2.4), we find

$$F_p = 0{\cdot}3 \pm 0{\cdot}1. \tag{2.7}$$

Clearly we do not claim for these calculations the high reliability usually accorded series estimates of critical parameters; instead the above values of g_p, A_p and F_p should be regarded as order of magnitude estimates only. The quite large value for F_p, as compared to the corresponding Ising amplitude F (Gaunt and Sykes 1972), is consistent with our assumption that the slow convergence associated with the basic series can be attributed solely to the confluent singularity.

Finally, we have used the standard techniques (Gaunt and Sykes 1972, Gaunt and Guttmann 1974) to estimate .the amplitude E_p of the leading singularity assuming $\delta_p = 18$. The residues at the pole close to $\lambda = 1$ of the Padé approximants to the $P_c^{-\delta_p}$ series have proved the most useful, but convergence is again rather slow by any of the methods. We find the following order of magnitude estimates:

$$E_p = \begin{cases} 1{\cdot}095 \pm 0{\cdot}005 & \text{HC(B)} \\ 1{\cdot}096 \pm 0{\cdot}002 & \text{SQ(B)} \\ 1{\cdot}100 \pm 0{\cdot}008 & \text{T(B)} \end{cases} \tag{2.8}$$

and

$$E_p = \begin{cases} 1{\cdot}08 \pm 0{\cdot}015 & \text{HC(S)} \\ 1{\cdot}09 \pm 0{\cdot}025 & \text{SQ(S)} \\ 1{\cdot}104 \pm 0{\cdot}006 & \text{T(S)} \\ 1{\cdot}105 \pm 0{\cdot}025 & \text{SQM(S)} \\ 1{\cdot}11 \pm 0{\cdot}02 & \text{HCM(S).} \end{cases} \tag{2.9}$$

A satisfactory feature of these values is their monotonic variation with lattice coordination number. The large uncertainties for the site problems (with the exception of the triangular lattice) allow for the uncertainties in p_c. Taking the central value of p_c for these lattices leads to uncertainties in E_p of the same order as for the triangular lattice.

3. Conclusions

We have analysed series expansions for the in-field percolation probability at the critical probability p_c. As the field ξ approaches zero (or $\lambda \to 1-$), the dominant asymptotic behaviour of $P_c(\lambda)$ is described by (1.8) with amplitude E_p given by (2.8) or (2.9). The critical exponent δ_p appears to be a dimensional invariant with a value given by (2.1). Earlier work by Essam and Gwilym (1971) based upon shorter series gave $\delta_p \geqslant 10$, while an analysis by Stauffer (1975) of extant Monte Carlo results indicates $1/\delta_p = 0{\cdot}0 \pm 0{\cdot}1$. More recently, Enting has considered the q-state random cluster model (Fortuin and Kasteleyn 1972) for which a $q \to 1$ limit is equivalent in zero field to the bond percolation problem. It should be noted that the field is introduced into the random cluster model in a manner different to (1.1) but this is not expected to affect the critical exponent. Preliminary series estimates by Enting (private communication) indicate a value consistent with (2.1) only with larger uncertainties (due to shorter series).

Our estimate of δ_p is in good agreement with that predicted from the scaling law (Essam and Gwilym 1971)

$$\delta_p = 1 + (\gamma_p/\beta_p) \tag{3.1}$$

using series estimates of γ_p and β_p. For the mean size exponent we have from II

$$\gamma_p = 2{\cdot}43 \pm 0{\cdot}03 \tag{3.2}$$

and for the percolation probability exponent

$$\beta_p = 0{\cdot}138 \pm 0{\cdot}007 \tag{3.3}$$

from IV. Substituting into (3.1) gives the scaling prediction

$$\delta_p = 18{\cdot}6 \pm 1{\cdot}2 \tag{3.4}$$

as compared to the direct series estimate (2.1).

It is our present opinion that

$$\gamma_p = 2\tfrac{3}{7}, \qquad \beta_p = \tfrac{1}{7}, \qquad \delta_p = 18 \tag{3.5}$$

is the simplest set of rational exponents which is most consistent with all the available data and which satisfy the scaling law (3.1) exactly. It will be interesting to see if the three-dimensional data are consistent with δ_p being an even integer, as it seems to be in two dimensions and is for the Bethe lattice for which $\delta_p = 2$ (Essam and Gwilym 1971). A similar result—but with odd integers—holds for the Ising model where apparently $\delta = 15, 5$ and 3 for two-dimensional, three-dimensional and Bethe lattices, respectively (Gaunt 1967, Gaunt and Sykes 1972).

According to scaling theory (Essam and Gwilym 1971, Dunn *et al* 1975a)

$$\Delta_p = \beta_p + \gamma_p, \qquad \nu_p = (2\beta_p + \gamma_p)/d \tag{3.6}$$

where Δ_p and ν_p are 'constant gap' exponents for the moments of the cluster size distribution and the pair connectivity, respectively, and d is the dimensionality. Using the numerical estimates (3.2) and (3.3) we obtain the scaling predictions

$$\Delta_p = 2{\cdot}57 \pm 0{\cdot}04, \qquad \nu_p = 1{\cdot}35 \pm 0{\cdot}02, \tag{3.7}$$

which are in good agreement with the direct series estimates (Essam *et al* 1976, Dunn *et al* 1975b)

$$\Delta_p = 2{\cdot}62 \pm 0{\cdot}08, \qquad \nu_p = 1{\cdot}34 \pm 0{\cdot}02. \tag{3.8}$$

We have chosen to calculate Δ_p and ν_p from the pair (β_p, γ_p) rather than (β_p, δ_p) or (γ_p, δ_p), since the uncertainties in the predicted exponents are found to be the smallest in this case. Note that the simple rational exponents (3.5) lead to

$$\Delta_p = 2\tfrac{4}{7} = 2{\cdot}5714\ldots, \qquad \nu_p = 1\tfrac{5}{14} = 1{\cdot}3571\ldots \tag{3.9}$$

which lie well within the uncertainties of the direct estimates (3.8).

Essam *et al* (1976) and Dunn *et al* (1975b) have pointed out that their direct series estimates of γ_p, ν_p and Δ_p are consistent with γ/ν, Δ/ν and γ/Δ having the same value for percolation and the Ising model. If correct, this result would be a further example of 'new' or 'weak' universality (Suzuki 1974). However, we would then expect $\delta_p = \delta$ which we have shown to be most unlikely. Even if the uncertainties in our estimate (2.1) are over-optimistic, it is very difficult to believe that the plots shown in figure 1 approach 15 as $n \to \infty$.

We have seen that the dominant singularity (1.8) is modified by a confluent correction term of the form (2.2) with a fairly large amplitude F_p given by (2.7). The correction exponent g_p is probably a dimensional invariant with a value close to (2.5). As shown below, such a value is not in accord with the analogous quantity for the simple Ising model (Gaunt and Sykes 1972). From Domb's generalized equation of state (Domb 1971, Domb and Gaunt 1971)

$$g = 1/\beta\delta \tag{3.10}$$

for the Ising model. However, as shown by Gaunt and Baker (1970), the amplitude of this correction term is probably zero—certainly very small. The next correction term has non-zero amplitude and corresponds to

$$g = 1 - (1/\delta). \tag{3.11}$$

Using (3.5) to calculate the analogous exponents for the percolation problem, we find $g_p = \frac{7}{18} = 0{\cdot}388\ldots$ and $g_p = \frac{17}{18} = 0{\cdot}944\ldots$ corresponding to (3.10) and (3.11) respectively. Both of these values seem ruled out by the series estimate (2.5). An adequate explanation of this exponent's value must await further theoretical developments.

Acknowledgment

This research has been supported by a grant from the Science Research Council.

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